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Laminated composite tubes under extension, torsion, bending, shearing and pressuring: a state space approach

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Abstract

We present a state space approach to extension, torsion, bending, shearing and pressuring of laminated composite tubes. One of the novel features is that we have formulated the basic equations of anisotropic elasticity in the cylindrical coordinate system into a state equation by a judicious arrangement of the displacement and stress variables so that the system matrix is independent of r. The formulation suggests a systematic way using matrix algebra and the transfer matrix to determine the stress and deformation in a multilayered cylindrically anisotropic tube under applied loads that do not vary in the axial direction. An exact analysis of the tube subjected to uniform surface tractions, an axial force, a torque and bending moments is presented. The solution consists of an axisymmetric state due to extension, torsion, uniform pressuring and shearing, and an asymmetric state due to bending. The formalism indicates that extension, torsion and pressuring interact; uniform shearing causes pure shears in the laminated tube, regardless of the number of layers. These deformations are uncoupled with bending of the tube.

Keywords: Laminated composite tube; Cylindrical anisotropy; Bending; Extension; Torsion; State space; Transfer matrix

1. Introduction

The circular cylindrical tube is a structural form frequently used in practice, especially in the offshore engineering and infrastructure. The tubular hybrid composites are useful in drilling operations and orbiting space structures. Analytical solutions of the deformation and stress in laminated tubes are of theoretical interest and practical importance. The solution may serve as a guide in designing riser tubes and tubular specimens. Also, it may be applied to many types of conductors with layers of protection and insulation.

When the tube is subjected to extension and uniform pressuring and shearing, it is in the state of *generalized plane strain*; when subjected to a torque at the ends and free from surface tractions and body forces, it is in the state of *generalized torsion* (Lekhnitskii, 1981). Analysis of these problems of an anisotropic tube is usually based on the Lekhnitskii stress function approach. While the stresses are determined from the stress functions by differentiation in the Lekhnitskii formalism, the displacements cannot be expressed by the stress

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functions in simple terms. Thus the formalism is ineffective for problems of laminates in which the conditions of interfacial continuity require the displacement as well as the traction be continuous. Extension of the Lekhnitskii formalism to layered cylinders has been done (Jolicoeur and Cardou, 1994; Chouchaoui and Ochoa, 1999), in which the solution expressions for each layer are determined first, and satisfaction of the interfacial continuity conditions and boundary conditions is then enforced. The solution scheme is straightforward, but due to the layerwise treatment, one has to deal with a very large system of equations for the undetermined constants in the stress and displacement expressions. Alternatively, one could use the displacement approach by deriving the governing equations in terms of displacements and seeking for the solution. The stress expressions in terms of the displacements now become very complicated. When applied to a multilayered system, one also has to confront the heavy task of dealing with a large system of equations. Using the displacement approach, Pagano (1972) solved the problem of a *homogeneous*, cylindrically anisotropic hollow circular cylinder under 2D surface tractions. Kollár and Springer (1992) and Kollár et al. (1992) presented a stress analysis of anisotropic laminated cylinders subjected to hygrothermal and mechanical loads. Indeed the displacement approach lead to unwieldy stress expressions and intricate solutions. The problem of a cylindrically anisotropic circular tube subjected to pressuring, shearing, torsion and extension was also studied by Ting (1996, 1999) and Chen et al. (2000). Their works were restricted to the *axisymmetric* deformation of a *homogeneous* tube in which the stress depends on r only.

In view of the drawbacks of using the stress or the displacement alone as the primary variables, it is sensible to formulate the problem in a state space framework in which the stress as well as the displacement are the state variables. Herein we develop a state space approach to the problem of multilayered cylindrically anisotropic tubes subjected to tractions that do not vary axially. Cylindrical anisotropy is not uncommon in cylindrical bodies, for examples, it appears in natural bamboo, tree trunk and carbon fiber (Christensen, 1994). The metallic forming process, such as extrusion or drawing, may result in cylindrically anisotropic products. The filamentary wound composite is a cylindrically orthotropic material on the macroscopic scale. To model the laminated composite tube produced by filament winding, we consider that the tube is composed of cylindrically monoclinic anisotropic layers. The cylindrical orthotropy is included as a special case. In a state space formulation an important step is to express the field equations in a state equation in which the unknown is the state vector. For problems of laminated tubes it is natural to take the displacements u_r, u_θ, u_z and the transverse stresses σ_r, σ_{rz}, $\sigma_{r\theta}$ as the primary state variables because the interfacial continuity conditions and the boundary conditions are directly associated with them. However, the field equations in the cylindrical coordinates are much more complicated than those in the Cartesian coordinates (Wang et al., 2000). If special arrangements are not made, the system matrix is inevitably r-dependent, making the state equation unsolvable by means of matrix algebra. To avoid this situation, we judiciously take $r\sigma_r$, $r\sigma_{rz}$, $r\sigma_{r\theta}$ instead of σ_r, σ_{rz}, $\sigma_{r\theta}$ as the stress variables and cast the field equations into a first order matrix equation with respect to r. It turned out that the system matrix is then independent of r so that it is possible to determine the solution for the laminated tube using methods of matrix algebra in conjunction with the transfer matrix. The transfer matrix transmits the state variable vector from the inner surface to the outer surface and takes into account the interfacial continuity and lateral boundary conditions in a simple manner. Its determination requires only matrix operation and eigensolutions of 6×6 matrices regardless of the number of layers.

In this paper we develop the state space formalism to treat the generalized plane strain, generalized torsion and bending problems of laminated composite tubes. An exact analysis of the tube subjected to uniform tractions on the inner and outer surfaces, and an axial force, a torque and bending moments at the ends is presented. To simplify the operation the characteristics of the eigensolution of the system matrix are used to advantage in deriving the fundamental transfer matrices that are essential for the analysis. The approach is verified by applying it to a cylindrically anisotropic homogeneous tube. The exact solutions of the tube under pressuring and bending in anisotropic elasticity are reproduced. It is further examined by a numerical example on bending of a laminated composite tube. Numerical results on the displacements and stresses through the thickness are computed following the solution procedure using *Mathematica* (Wolfram, 1996).

2. State space formulation

2.1. Problem statement

Consider a circular tube composed of n anisotropic layers as shown in Fig. 1. Referred to the cylindrical coordinates (r, θ, z), the material is cylindrically anisotropic having reflectional symmetry with respect to the cylindrical surfaces $r =$ constant at each point. The stress–displacement relations of the material are (Lekhnitskii, 1981)

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix}_k = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{13} & c_{23} & c_{33} & c_{34} & 0 & 0 \\ c_{14} & c_{24} & c_{34} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{56} & c_{66} \end{bmatrix}_k \begin{bmatrix} u_{r,r} \\ r^{-1}(u_{\theta,\theta} + u_r) \\ u_{z,z} \\ u_{\theta,z} + r^{-1}u_{z,\theta} \\ u_{z,r} + u_{r,z} \\ r^{-1}u_{r,\theta} + u_{\theta,r} - r^{-1}u_\theta \end{bmatrix}_k, \tag{1}$$

where $\sigma_r, \sigma_\theta, \ldots, \sigma_{r\theta}$ are the stress components; u_r, u_θ, u_z are the displacements; c_{ij} are the 13 elastic constants of the cylindrically monoclinic anisotropic material; a comma denotes partial differentiation with respect to the suffix variables; the subscript k denotes the kth layers. Henceforth, k runs from 1 to n unless indicated otherwise.

An important class of the material under consideration is the fiber-reinforced composite produced by filament winding, which may be regarded as a cylindrically orthotropic material with the fiber direction oriented by a helix angle to the z-axis. In this case the 13 constants c_{ij} in Eq. (1) are derivable from nine independent ones by a rotation about the radial axis.

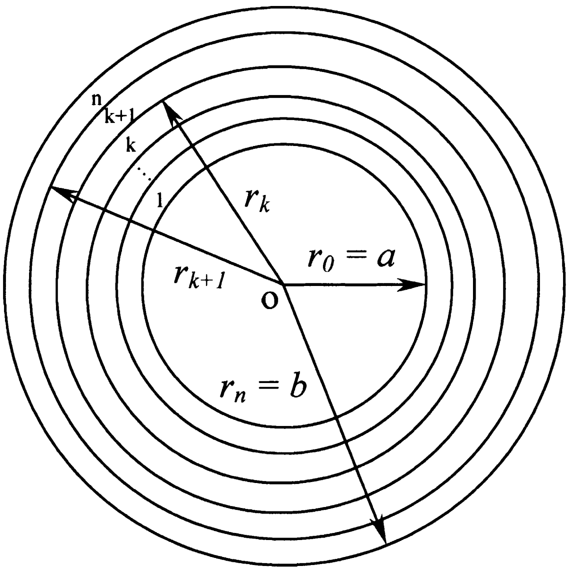

Fig. 1. A multilayered laminated composite tube.

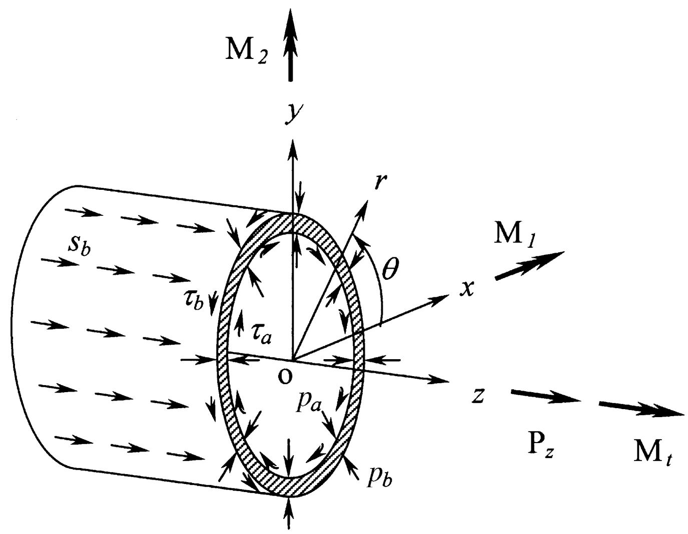

Fig. 2. A schematic configuration of the tube subjected to uniform tractions and end loads.

When the tube is subjected to the end loads and surface tractions that do not vary in the z-axis, as shown in Fig. 2, the stress is independent of z . The basic equations of equilibrium in the cylindrical coordinates (Lekhnitskii, 1981) are

$$\sigma_{r,r} + r^{-1}\sigma_{r\theta,\theta} + r^{-1}(\sigma_r - \sigma_\theta) + R = 0, \tag{2}$$

$$\sigma_{r\theta,r} + r^{-1}\sigma_{\theta,\theta} + 2r^{-1}\sigma_{r\theta} + \Theta = 0, \tag{3}$$

$$\sigma_{rz,r} + r^{-1}\sigma_{\theta z,\theta} + r^{-1}\sigma_{rz} = 0, \tag{4}$$

where R, Θ denote the body forces in the coordinate directions r, θ.

The boundary conditions on the inner and outer surfaces are

$$[\sigma_r \quad \sigma_{r\theta} \quad \sigma_{rz}]_1 = [-p_a \quad \tau_a \quad s_a] \quad \text{on } r = a, \tag{5}$$

$$[\sigma_r \quad \sigma_{r\theta} \quad \sigma_{rz}]_n = [-p_b \quad \tau_b \quad s_b] \quad \text{on } r = b, \tag{6}$$

where p_a, p_b are the internal and external pressure; τ_a, τ_b are the uniform in-plane shears, s_a, s_b the uniform anti-plane shears. For static equilibrium the prescribed tractions must satisfy the conditions $\tau_a a^2 = \tau_b b^2$ and $s_a a = s_b b$.

The end conditions require that the stress resultants reduce to an axial force P_z, a torque M_t, and bi-axial bending moments M_1, M_2, such that

$$\sum_{k=1}^{n} \int_0^{2\pi} \int_{r_{k-1}}^{r_k} (r\sigma_z)_k \,\mathrm{d}r\,\mathrm{d}\theta = P_z, \tag{7}$$

$$\sum_{k=1}^{n} \int_0^{2\pi} \int_{r_{k-1}}^{r_k} (r\sigma_{\theta z})_k r \,\mathrm{d}r\,\mathrm{d}\theta = M_t, \tag{8}$$

$$\sum_{k=1}^{n} \int_{0}^{2\pi} \int_{r_{k-1}}^{r_k} (r\sigma_z)_k r \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta = M_1, \tag{9}$$

$$\sum_{k=1}^{n} \int_{0}^{2\pi} \int_{r_{k-1}}^{r_k} (r\sigma_z)_k r \cos\theta \,\mathrm{d}r\,\mathrm{d}\theta = M_2, \tag{10}$$

where r_{k-1} and r_k denote the internal and external radii of the kth layer, thus $r_0 = a$, $r_n = b$.

The conditions that the resultant shears vanish at the ends are satisfied identically when the stress is independent of z (see proof in Appendix A). We note that Eqs. (7)–(10) are the resultant form of the traction boundary conditions. Using them implies that the end effect is neglected.

The interfacial continuity conditions require

$$[u_r \quad u_\theta \quad u_z \quad \sigma_r \quad \sigma_{r\theta} \quad \sigma_{rz}]_{k+1} = [u_r \quad u_\theta \quad u_z \quad \sigma_r \quad \sigma_{r\theta} \quad \sigma_{rz}]_k, \tag{11}$$

on $r = r_k$ for $k = 1, 2, \ldots, n-1$.

For the problem under study the stress is independent of z, but the displacement may depend on z. The general expressions for the displacement field (Lekhnitskii, 1981) are

$$u_r = u(r,\theta) - \frac{z^2}{2}(A\cos\theta + B\sin\theta) + z(\omega_2\cos\theta - \omega_1\sin\theta) + u_0\cos\theta + v_0\sin\theta, \tag{12}$$

$$u_\theta = v(r,\theta) + \frac{z^2}{2}(A\sin\theta - B\cos\theta) + \vartheta rz - z(\omega_2\sin\theta + \omega_1\cos\theta) + \omega_3 r - u_0\sin\theta + v_0\cos\theta, \tag{13}$$

$$u_z = w(r,\theta) + z(Ar\cos\theta + Br\sin\theta + \varepsilon) + r(\omega_1\sin\theta - \omega_2\cos\theta) + w_0, \tag{14}$$

where u, v, w are unknown functions of r and θ; u_0, v_0, w_0, ω_1, ω_2, ω_3 are constants characterizing the rigid body displacements; the constants ε is a uniform extension, ϑ is the twisting angle per unit length along z-axis, A and B are associated with bending of the tube. In order to satisfy the interfacial continuity conditions, it is necessary that these constants be the same for all layers.

2.2. State equation

Derivation of the state equation based on Eqs. (2)–(4) as they stand will result in a system matrix with *variable* coefficients, not solvable using matrix algebra (Pease, 1965). Aiming at formulating the field equations into a system of first order differential equations with respect to r, we rearrange Eqs. (2)–(4) as

$$(r\sigma_r)_{,r} + \sigma_{r\theta,\theta} - \sigma_\theta + rR = 0, \tag{15}$$

$$(r\sigma_{r\theta})_{,r} + \sigma_{\theta,\theta} + \sigma_{r\theta} + r\Theta = 0, \tag{16}$$

$$(r\sigma_{rz})_{,r} + \sigma_{\theta z,\theta} = 0. \tag{17}$$

The transverse stresses with respect to r are much more concise in Eqs. (15)–(17) than in Eqs. (2)–(4). This prompts us to take $r\sigma_r$, $r\sigma_{r\theta}$, $r\sigma_{rz}$ instead of σ_r, $\sigma_{r\theta}$, σ_{rz} as the stress variables in deriving the state equation. Thus, taking u_r, u_θ, u_z, $r\sigma_r$, $r\sigma_{r\theta}$, $r\sigma_{rz}$ to form the state vector and expressing $r\sigma_\theta$, $r\sigma_z$, $r\sigma_{\theta z}$ in terms of them, we derive in Appendix B the matrix differential equation:

$$\frac{\partial}{\partial r}\begin{bmatrix} u_r \\ u_\theta \\ u_z \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz} \end{bmatrix} = r^{-1}\begin{bmatrix} -\hat{c}_{12} & d_{12} & d_{13} & c_{11}^{-1} & 0 & 0 \\ -\partial_\theta & 1 & 0 & 0 & s_{55} & s_{56} \\ -r\partial_z & 0 & 0 & 0 & s_{56} & s_{66} \\ Q_{22} & d_{42} & d_{43} & \hat{c}_{12} & -\partial_\theta & 0 \\ -Q_{22}\partial_\theta & d_{52} & d_{53} & -\hat{c}_{12}\partial_\theta & -1 & 0 \\ -Q_{24}\partial_\theta & d_{62} & d_{63} & -\hat{c}_{14}\partial_\theta & 0 & 0 \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \\ u_z \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ rR \\ r\Theta \\ 0 \end{bmatrix}, \tag{18}$$

$$\begin{bmatrix} r\sigma_\theta \\ r\sigma_z \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} Q_{22} & Q_{22}\partial_\theta + Q_{24}r\partial_z & Q_{24}\partial_\theta + Q_{23}r\partial_z \\ Q_{23} & Q_{23}\partial_\theta + Q_{34}r\partial_z & Q_{34}\partial_\theta + Q_{33}r\partial_z \\ Q_{24} & Q_{24}\partial_\theta + Q_{44}r\partial_z & Q_{44}\partial_\theta + Q_{34}r\partial_z \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} + \begin{bmatrix} \hat{c}_{12} \\ \hat{c}_{13} \\ \hat{c}_{14} \end{bmatrix} r\sigma_r, \tag{19}$$

where ∂_θ, ∂_z denote partial derivatives with respect to the suffix variables, and

$$d_{12} = -(\hat{c}_{12}\partial_\theta + \hat{c}_{14}r\partial_z), \quad d_{13} = -(\hat{c}_{14}\partial_\theta + \hat{c}_{13}r\partial_z), \quad d_{42} = Q_{22}\partial_\theta + Q_{24}r\partial_z,$$

$$d_{43} = Q_{24}\partial_\theta + Q_{23}r\partial_z, \quad d_{52} = -\partial_\theta(Q_{22}\partial_\theta + Q_{24}r\partial_z), \quad d_{53} = -\partial_\theta(Q_{24}\partial_\theta + Q_{23}r\partial_z),$$

$$d_{62} = -\partial_\theta(Q_{24}\partial_\theta + Q_{44}r\partial_z), \quad d_{63} = -\partial_\theta(Q_{44}\partial_\theta + Q_{34}r\partial_z),$$

$$\hat{c}_{ij} = c_{ij}/c_{11}, \quad Q_{ij} = c_{ij} - c_{1i}c_{1j}/c_{11}, \quad \begin{bmatrix} s_{55} & s_{56} \\ s_{56} & s_{66} \end{bmatrix} = \frac{1}{c_{55}c_{66} - c_{56}^2}\begin{bmatrix} c_{66} & -c_{56} \\ -c_{56} & c_{55} \end{bmatrix}.$$

On substituting Eqs. (12)–(14) in Eqs. (18) and (19), these equations in the absence of the body forces become

$$r\frac{\partial}{\partial r}\begin{bmatrix} u \\ v \\ w \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz} \end{bmatrix} = \begin{bmatrix} -\hat{c}_{12} & -\hat{c}_{12}\partial_\theta & -\hat{c}_{14}\partial_\theta & c_{11}^{-1} & 0 & 0 \\ -\partial_\theta & 1 & 0 & 0 & s_{55} & s_{56} \\ 0 & 0 & 0 & 0 & s_{56} & s_{66} \\ Q_{22} & Q_{22}\partial_\theta & Q_{24}\partial_\theta & \hat{c}_{12} & -\partial_\theta & 0 \\ -Q_{22}\partial_\theta & -Q_{22}\partial_{\theta\theta} & -Q_{24}\partial_{\theta\theta} & -\hat{c}_{12}\partial_\theta & -1 & 0 \\ -Q_{24}\partial_\theta & -Q_{24}\partial_{\theta\theta} & -Q_{44}\partial_{\theta\theta} & -\hat{c}_{14}\partial_\theta & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz} \end{bmatrix} + Ar^2\begin{bmatrix} -\hat{c}_{13}\cos\theta \\ 0 \\ 0 \\ Q_{23}\cos\theta \\ Q_{23}\sin\theta \\ Q_{34}\sin\theta \end{bmatrix}$$
$$+ Br^2\begin{bmatrix} -\hat{c}_{13}\sin\theta \\ 0 \\ 0 \\ Q_{23}\sin\theta \\ -Q_{23}\cos\theta \\ -Q_{34}\cos\theta \end{bmatrix} + \varepsilon r\begin{bmatrix} -\hat{c}_{13} \\ 0 \\ 0 \\ Q_{23} \\ 0 \\ 0 \end{bmatrix} + \vartheta r^2\begin{bmatrix} 0 \\ 0 \\ 0 \\ Q_{24} \\ 0 \\ 0 \end{bmatrix}, \tag{20}$$

$$\begin{bmatrix} r\sigma_\theta \\ r\sigma_z \\ r\sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} Q_{22} & Q_{22}\partial_\theta & Q_{24}\partial_\theta \\ Q_{23} & Q_{23}\partial_\theta & Q_{34}\partial_\theta \\ Q_{24} & Q_{24}\partial_\theta & Q_{44}\partial_\theta \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} \hat{c}_{12} \\ \hat{c}_{13} \\ \hat{c}_{14} \end{bmatrix} r\sigma_r + (Ar^2\cos\theta + Br^2\sin\theta + \varepsilon r)\begin{bmatrix} Q_{23} \\ Q_{33} \\ Q_{34} \end{bmatrix} + \vartheta r^2\begin{bmatrix} Q_{24} \\ Q_{34} \\ Q_{44} \end{bmatrix}. \tag{21}$$

In Eq. (20) r^{-1} is a common factor of the system matrix and has been taken out to the left hand side so that the system matrix is independent of r. This makes it possible to solve the state equation using matrix algebra and the transfer matrix—it could not have been done without taking $r\sigma_r$, $r\sigma_{r\theta}$, $r\sigma_{rz}$ to be the stress variables.

3. Solution using transfer matrix

3.1. Solution to the state equation

A close examination of Eq. (20) led us to assume the solution in the form

$$\begin{bmatrix} u \\ v \\ w \\ r\sigma_r \\ r\sigma_{r\theta} \\ r\sigma_{rz} \end{bmatrix} = \begin{bmatrix} U_1(r) \\ V_1(r) \\ W_1(r) \\ X_1(r) \\ Y_1(r) \\ Z_1(r) \end{bmatrix} + \begin{bmatrix} U_2(r)\cos\theta \\ V_2(r)\sin\theta \\ W_2(r)\sin\theta \\ X_2(r)\cos\theta \\ Y_2(r)\sin\theta \\ Z_2(r)\sin\theta \end{bmatrix} + \begin{bmatrix} U_3(r)\sin\theta \\ V_3(r)\cos\theta \\ W_3(r)\cos\theta \\ X_3(r)\sin\theta \\ Y_3(r)\cos\theta \\ Z_3(r)\cos\theta \end{bmatrix}, \tag{22}$$

in which $U_1, V_1, \ldots, Z_3$ are three sets of unknown functions of r. The part with a subscript 1 represents an axisymmetric state due to extension, torsion, uniform shearing and pressuring. The other parts are useful for the asymmetric state due to bending.

Substituting Eq. (22) in Eq. (20) yields three sets of uncoupled systems of first order ordinary differential equations. Written in a compact form, they are

(1)

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} \mathbf{U}_1 \\ \mathbf{S}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{13} & -\mathbf{N}_{11}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{U}_1 \\ \mathbf{S}_1 \end{bmatrix} + \varepsilon r\begin{bmatrix} \boldsymbol{\phi}_{u1} \\ \boldsymbol{\phi}_{s1} \end{bmatrix} + \vartheta r^2\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\varphi}_{s1} \end{bmatrix}, \tag{23}$$

$$\begin{bmatrix} r\sigma_\theta^{(1)} \\ r\sigma_z^{(1)} \\ r\sigma_{\theta z}^{(1)} \end{bmatrix} = \begin{bmatrix} Q_{22} \\ Q_{23} \\ Q_{24} \end{bmatrix} U_1 + \begin{bmatrix} \hat{c}_{12} \\ \hat{c}_{13} \\ \hat{c}_{14} \end{bmatrix} X_1 + \varepsilon r\begin{bmatrix} Q_{23} \\ Q_{33} \\ Q_{34} \end{bmatrix} + \vartheta r^2\begin{bmatrix} Q_{24} \\ Q_{34} \\ Q_{44} \end{bmatrix}, \tag{24}$$

where

$$\mathbf{U}_1 = [U_1 \quad V_1 \quad W_1]^{\mathrm{T}}, \quad \mathbf{S}_1 = [X_1 \quad Y_1 \quad Z_1]^{\mathrm{T}}, \quad \boldsymbol{\phi}_{u1} = \begin{bmatrix} -\hat{c}_{13} & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \boldsymbol{\phi}_{s1} = [Q_{23} \quad 0 \quad 0]^{\mathrm{T}},$$

$$\boldsymbol{\varphi}_{s1} = [Q_{24} \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{N}_{11} = \begin{bmatrix} -\hat{c}_{12} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_{12} = \mathbf{N}_{12}^{\mathrm{T}} = \begin{bmatrix} c_{11}^{-1} & 0 & 0 \\ 0 & s_{55} & s_{56} \\ 0 & s_{56} & s_{66} \end{bmatrix},$$

$$\mathbf{N}_{13} = \mathbf{N}_{13}^{\mathrm{T}} = \begin{bmatrix} Q_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(2)

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} \mathbf{U}_2 \\ \mathbf{S}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{21} & \mathbf{N}_{22} \\ \mathbf{N}_{23} & -\mathbf{N}_{21}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{U}_2 \\ \mathbf{S}_2 \end{bmatrix} + Ar^2\begin{bmatrix} \boldsymbol{\phi}_{u2} \\ \boldsymbol{\phi}_{s2} \end{bmatrix}, \tag{25}$$

$$\begin{bmatrix} r\sigma_\theta^{(2)} \\ r\sigma_z^{(2)} \\ r\sigma_{\theta z}^{(2)} \end{bmatrix} = \left(\begin{bmatrix} Q_{22} & Q_{22} & Q_{24} \\ Q_{23} & Q_{23} & Q_{34} \\ Q_{24} & Q_{24} & Q_{44} \end{bmatrix}\begin{bmatrix} U_2 \\ V_2 \\ W_2 \end{bmatrix} + \begin{bmatrix} \hat{c}_{12} \\ \hat{c}_{13} \\ \hat{c}_{14} \end{bmatrix} X_2 + Ar^2\begin{bmatrix} Q_{23} \\ Q_{33} \\ Q_{34} \end{bmatrix}\right)\cos\theta, \tag{26}$$

where

$$\mathbf{U}_2 = [U_2 \quad V_2 \quad W_2]^{\mathrm{T}}, \quad \mathbf{S}_2 = [X_2 \quad Y_2 \quad Z_2]^{\mathrm{T}}, \quad \boldsymbol{\phi}_{u2} = \begin{bmatrix} -\hat{c}_{13} & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \boldsymbol{\phi}_{s2} = [Q_{23} \quad Q_{23} \quad Q_{34}]^{\mathrm{T}},$$

$$\mathbf{N}_{21} = \begin{bmatrix} -\hat{c}_{12} & -\hat{c}_{12} & -\hat{c}_{14} \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_{22} = \mathbf{N}_{12}, \quad \mathbf{N}_{23} = \mathbf{N}_{23}^{\mathrm{T}} = \begin{bmatrix} Q_{22} & Q_{22} & Q_{24} \\ Q_{22} & Q_{22} & Q_{24} \\ Q_{24} & Q_{24} & Q_{44} \end{bmatrix}.$$

(3)

$$r\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} \mathbf{U}_3 \\ \mathbf{S}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{31} & \mathbf{N}_{32} \\ \mathbf{N}_{33} & -\mathbf{N}_{31}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{U}_3 \\ \mathbf{S}_3 \end{bmatrix} + Br^2\begin{bmatrix} \boldsymbol{\phi}_{u3} \\ \boldsymbol{\phi}_{s3} \end{bmatrix}, \tag{27}$$

$$\begin{bmatrix} r\sigma_\theta^{(3)} \\ r\sigma_z^{(3)} \\ r\sigma_{\theta z}^{(3)} \end{bmatrix} = \left(\begin{bmatrix} Q_{22} & -Q_{22} & -Q_{24} \\ Q_{23} & -Q_{23} & -Q_{34} \\ Q_{24} & -Q_{24} & -Q_{44} \end{bmatrix}\begin{bmatrix} U_3 \\ V_3 \\ W_3 \end{bmatrix} + \begin{bmatrix} \hat{c}_{12} \\ \hat{c}_{13} \\ \hat{c}_{14} \end{bmatrix} X_3 + Br^2\begin{bmatrix} Q_{23} \\ Q_{33} \\ Q_{34} \end{bmatrix} \right) \sin\theta, \tag{28}$$

where

$$\mathbf{U}_3 = [U_3 \quad V_3 \quad W_3]^{\mathrm{T}}, \quad \mathbf{S}_3 = [X_3 \quad Y_3 \quad Z_3]^{\mathrm{T}}, \quad \boldsymbol{\phi}_{u3} = [-\hat{c}_{13} \quad 0 \quad 0]^{\mathrm{T}}, \quad \boldsymbol{\phi}_{s3} = [Q_{23} \quad -Q_{23} \quad -Q_{34}]^{\mathrm{T}},$$

$$\mathbf{N}_{31} = \begin{bmatrix} -\hat{c}_{12} & \hat{c}_{12} & \hat{c}_{14} \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_{32} = \mathbf{N}_{12}, \quad \mathbf{N}_{33} = \mathbf{N}_{33}^{\mathrm{T}} = \begin{bmatrix} Q_{22} & -Q_{22} & -Q_{24} \\ -Q_{22} & Q_{22} & Q_{24} \\ -Q_{24} & Q_{24} & Q_{44} \end{bmatrix}.$$

At this stage it is easily shown that the end conditions (9) and (10) are satisfied identically by Eq. (24), leaving (7) and (8) for the ε and ϑ. Thus, ε and ϑ are related to P_z and M_t, unrelated to M_1 and M_2. The end conditions (7)–(9) are satisfied identically by Eq. (26), leaving (10) for the A so that A is related to M_2, unrelated to P_z and M_t. Similarly, B is related to M_1 via (9), unrelated to P_z and M_t.

Eqs. (23), (25) and (27) may be expressed in short as

$$r\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{X}(r) = \mathbf{A}\mathbf{X}(r) + \mathbf{f}(r). \tag{29}$$

To solve Eq. (29), let us introduce a change of variable

$$r = \mathrm{e}^x, \quad x = \log r, \tag{30}$$

to reduce it to

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{X}(x) = \mathbf{A}\mathbf{X}(x) + \mathbf{f}(\mathrm{e}^x). \tag{31}$$

Eq. (31) is a standard first-order matrix differential equation with a constant coefficient matrix (Pease, 1965), whose solution is

$$\mathbf{X}(x) = \mathbf{P}(x - x_0)\mathbf{X}(x_0) + \mathbf{q}(x), \tag{32}$$

where

$$\mathbf{P}(x - x_0) = \mathrm{e}^{\mathbf{A}(x - x_0)}, \tag{33}$$

$$\mathbf{q}(x) = \int_{x_0}^{x} \mathbf{P}(x - \eta)\mathbf{f}(\mathrm{e}^\eta)\,\mathrm{d}\eta. \tag{34}$$

The solution of Eq. (29) is obtained by replacing x by $\log r$ in Eqs. (32)–(34), yielding

$$\mathbf{X}(r) = \mathbf{P}(r/r_0)\mathbf{X}(r_0) + \mathbf{q}(r), \tag{35}$$

where

$$\mathbf{P}(r/r_0) = (r/r_0)^{\mathbf{A}}, \tag{36}$$

$$\mathbf{q}(r) = \int_{r_0}^{r} \xi^{-1}\mathbf{P}(r/\xi)\mathbf{f}(\xi)\,\mathrm{d}\xi. \tag{37}$$

The fundamental transfer matrix **P** given by Eq. (36) is a formal expression involving the function of a matrix. To express it in an operational form it is necessary to determine the eigensolution of **A**. One useful operational form is

$$\mathbf{P}(r/r_0) = (r/r_0)^{\mathbf{A}} = \mathbf{M}\langle (r/r_0)^{\lambda}\rangle \mathbf{M}^{-1}, \tag{38}$$

which is obtained by uncoupling Eq. (29) through diagonalization, where $\langle (r/r_0)^{\lambda}\rangle$ denotes the diagonal matrix consisting of the *distinct* eigenvalues λ associated with **A**. When **A** has repeated eigenvalues and is non-semi-simple (Pease, 1965), the Jordan matrix takes the place of the diagonal matrix. **M** is the modal matrix of **A**, consisting of the eigenvectors associated with the distinct eigenvalues and the *generalized eigenvectors* associated with the *repeated eigenvalues*.

To use Eq. (38) $\mathbf{M}^{-1}$ needs to be determined. This of course can be done by direct inversion but it takes a great deal of computation for a laminate composed of many layers. The inversion computation could be alleviated by making use of the orthogonality properties of the eigenvectors of **A**. For the sake of clarity we summarize the characteristics of the eigensolution of **A** in Appendix C.

3.2. Solution for a multilayered system

Eq. (29) can be written for each layer in a laminated tube as

$$r\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{X}_k(r) = \mathbf{A}_k\mathbf{X}_k(r) + \mathbf{f}_k(r), \quad r_{k-1} \leqslant r \leqslant r_k. \tag{39}$$

The solution to Eq. (39) is

$$\mathbf{X}_k(r) = \mathbf{P}_k(r/r_{k-1})\mathbf{X}_k(r_{k-1}) + \mathbf{q}_k(r), \tag{40}$$

where

$$\mathbf{P}_k(r/r_{k-1}) = (r/r_{k-1})^{\mathbf{A}_k}, \tag{41}$$

$$\mathbf{q}_k(r) = \int_{r_{k-1}}^{r} \xi^{-1}\mathbf{P}_k(r/\xi)\mathbf{f}_k(\xi)\,\mathrm{d}\xi. \tag{42}$$

The interfacial continuity conditions (11) are satisfied by letting

$$\mathbf{X}_{k+1}(r_k) = \mathbf{X}_k(r_k). \tag{43}$$

There follows

$$\mathbf{X}_{k+1}(r_k) = \mathbf{P}_k(r_k/r_{k-1})\mathbf{X}_k(r_{k-1}) + \mathbf{q}_k(r_k), \tag{44}$$

for $k = 1, 2, \ldots, n-1$.

On transferring the state vector from the inner surface outward, we obtain

$$\mathbf{X}(r) = \mathbf{T}_k(r)\mathbf{X}_1(a) + \boldsymbol{\phi}_k(r), \quad r_{k-1} \leqslant r \leqslant r_k, \tag{45}$$

where

$$\mathbf{T}_k(r) = \begin{cases} \mathbf{P}_1(r/a) & k = 1 \\ \mathbf{P}_k(r/r_{k-1})\mathbf{T}_{k-1}(r_{k-1}) & k = 2, 3, \ldots, n, \end{cases} \tag{46}$$

$$\boldsymbol{\phi}_k(r) = \begin{cases} \mathbf{q}_1(r) & k = 1 \\ \mathbf{P}_k(r/r_{k-1})\boldsymbol{\phi}_{k-1}(r_{k-1}) + \mathbf{q}_k(r) & k = 2, 3, \ldots, n. \end{cases} \tag{47}$$

Setting $r = b$ in Eq. (45) gives

$$\mathbf{X}(b) = \mathbf{T}_n(b)\mathbf{X}_1(a) + \boldsymbol{\phi}_n(b). \tag{48}$$

This relation connects the state vectors on $r = a$ and b where the boundary conditions (5) and (6) are prescribed. To facilitate satisfaction of Eqs. (5) and (6), let us express Eq. (48) as

$$\begin{bmatrix} \mathbf{U}_i(b) \\ \mathbf{S}_i(b) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{uu}(b) & \mathbf{T}_{us}(b) \\ \mathbf{T}_{su}(b) & \mathbf{T}_{ss}(b) \end{bmatrix} \begin{bmatrix} \mathbf{U}_i(a) \\ \mathbf{S}_i(a) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\phi}_{ui}(b) \\ \boldsymbol{\phi}_{si}(b) \end{bmatrix}. \tag{49}$$

The boundary conditions (5) and (6) demand

$$\mathbf{S}_1(a) = [\,-ap_a \quad a\tau_a \quad as_a\,], \quad \mathbf{S}_1(b) = [\,-bp_b \quad b\tau_b \quad bs_b\,], \tag{50}$$

$$\mathbf{S}_2(a) = \mathbf{S}_3(a) = 0, \quad \mathbf{S}_2(b) = \mathbf{S}_3(b) = 0. \tag{51}$$

Imposing Eqs. (50) and (51) on Eq. (49) and solving for the unknowns on $r = a$ yields

$$\mathbf{U}_1(a) = \mathbf{T}_{su}^{-1}(b)[\mathbf{S}_1(b) - \mathbf{T}_{ss}(b)\mathbf{S}_1(a) - \boldsymbol{\phi}_{s1}(b)], \tag{52}$$

$$\mathbf{U}_2(a) = -\mathbf{T}_{su}^{-1}(b)\boldsymbol{\phi}_{s2}(b), \tag{53}$$

$$\mathbf{U}_3(a) = -\mathbf{T}_{su}^{-1}(b)\boldsymbol{\phi}_{s3}(b). \tag{54}$$

Substituting Eqs. (50)–(54) in Eq. (45) gives us the displacement and transverse stress variables through the thickness:

$$\begin{bmatrix} \mathbf{U}_i(r) \\ \mathbf{S}_i(r) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{uu}(r) & \mathbf{T}_{us}(r) \\ \mathbf{T}_{su}(r) & \mathbf{T}_{ss}(r) \end{bmatrix} \begin{bmatrix} \mathbf{U}_i(a) \\ \mathbf{S}_i(a) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\phi}_{ui}(r) \\ \boldsymbol{\phi}_{si}(r) \end{bmatrix}. \tag{55}$$

It should be noted that uniform surface tractions enter the picture through the boundary conditions (49), resulting in an axisymmetric state in the tube. Uniform shearing and pressuring cause only axisymmetric deformation without bending.

4. Axisymmetric state

The applied loads that give rise to an axisymmetric state include an axial force and a torque at the ends, and the internal and external pressure, uniform in-plane and anti-plane shears on the inner and outer surfaces. They produce extension, torsion, radial contraction and circumferential deformation, but not bending of the tube.

The governing equation for the axisymmetric state is Eq. (23). Determination of the fundamental transfer matrix requires the eigensolution of $\mathbf{A}_k$. The eigenvalues of $\mathbf{A}_k$ are

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = (c_{22}/c_{11})^{1/2}, \quad \lambda_4 = 0, \quad \lambda_5 = -1, \quad \lambda_6 = -(c_{22}/c_{11})^{1/2}, \tag{56}$$

of which $\lambda_1 = 0$ is a repeated one, thus only one corresponding eigenvector $\boldsymbol{\varphi}_1$ is determined in the usual way. The *generalized eigenvector* $\boldsymbol{\varphi}_4$ is determined by means of the Jordan chain $\mathbf{A}\varphi_4 = \lambda_4\boldsymbol{\varphi}_4 + \boldsymbol{\varphi}_1$ with $\lambda_4 = 0$.

On determining the eigenvectors, the modal matrix is formed as

$$\mathbf{M}_k = \begin{bmatrix} 0 & 0 & \eta_1 & 0 & 0 & -\eta_1 \\ 0 & 1 & 0 & -s_{56}s_{66}^{-1/2} & -s_{55}/2 & 0 \\ s_{66}^{1/2} & 0 & 0 & 0 & -s_{56} & 0 \\ 0 & 0 & \eta_1\varkappa_1 & 0 & 0 & \eta_1\varkappa_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s_{66}^{-1/2} & 0 & 0 \end{bmatrix}_k, \tag{57}$$

where

$$\eta_1 = (4c_{11}c_{22})^{-1/4}, \quad \varkappa_1 = (c_{11}c_{22})^{1/2} + c_{12}, \quad \varkappa_2 = (c_{11}c_{22})^{1/2} - c_{12}.$$

By Eq. (C.3) in Appendix C we have

$$\mathbf{M}_k^{-1} = \begin{bmatrix} 0 & 0 & s_{66}^{-1/2} & 0 & s_{56}s_{66}^{-1/2} & 0 \\ 0 & 1 & 0 & 0 & s_{55}/2 & s_{56} \\ \eta_1\varkappa_2 & 0 & 0 & \eta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66}^{1/2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\eta_1\varkappa_1 & 0 & 0 & \eta_1 & 0 & 0 \end{bmatrix}_k. \tag{58}$$

It can be verified indeed $\mathbf{M}_k\mathbf{M}_k^{-1} = \mathbf{I}$. Substituting Eqs. (57) and (58) in Eq. (38), replacing the diagonal matrix by the Jordan matrix for the repeated eigenvalues $\lambda_1 = \lambda_4 = 0$, we obtain

$$\mathbf{P}(r/r_{k-1}) = \begin{bmatrix} p_{11} & 0 & 0 & p_{14} & 0 & 0 \\ 0 & r/r_{k-1} & 0 & 0 & s_{55}(r/r_{k-1} - r_{k-1}/r)/2 & -s_{56}(1 - r/r_{k-1}) \\ 0 & 0 & 1 & 0 & s_{56}(1 - r/r_{k-1}) & s_{66}\log(r/r_{k-1}) \\ p_{41} & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{k-1}/r & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_k, \tag{59}$$

where

$$\kappa = (c_{22}/c_{11})^{1/2},$$

$$p_{11} = [\varkappa_2(r/r_{k-1})^{\kappa} + \varkappa_1(r/r_{k-1})^{-\kappa}]/(2\kappa c_{11}), \qquad p_{14} = [(r/r_{k-1})^{\kappa} - (r/r_{k-1})^{-\kappa}]/(2\kappa c_{11}),$$

$$p_{41} = \varkappa_1\varkappa_2[(r/r_{k-1})^{\kappa} - (r/r_{k-1})^{-\kappa}]/(2\kappa c_{11}), \qquad p_{44} = [\varkappa_1(r/r_{k-1})^{\kappa} + \varkappa_2(r/r_{k-1})^{-\kappa}]/(2\kappa c_{11}).$$

The non-homogeneous terms in Eq. (23) consist of two parts:

$$\mathbf{f}_k(r) = r\mathbf{g}_1 = r\varepsilon[-c_{13}/c_{11} \quad 0 \quad 0 \quad Q_{23} \quad 0 \quad 0]_k^{\mathrm{T}}, \tag{60}$$

and

$$\mathbf{f}_k(r) = r^2\mathbf{g}_2 = r^2\vartheta[0 \quad 0 \quad 0 \quad Q_{24} \quad 0 \quad 0]_k^{\mathrm{T}}. \tag{61}$$

For $\mathbf{f}_k(r) = r\mathbf{g}_1$ the matrix $\mathbf{A}_k - \mathbf{I}$ is *singular* so that $(\mathbf{A}_k - \mathbf{I})^{-1}$ does not exist. When $c_{22} \neq c_{11}$, the vector annihilated by $\mathbf{A}_k^{\mathrm{T}} - \mathbf{I}$ is $[0 \quad 1 \quad 0 \quad 0 \quad s_{55}/2 \quad s_{56}]_k^{\mathrm{T}}$ which is orthogonal with $\mathbf{g}_1$. It can be shown that $(\mathbf{A}_k - \mathbf{I})\mathbf{z}_1 = \mathbf{g}_1$ has the solution

$$\mathbf{z}_1 = \frac{\varepsilon}{c_{11} - c_{22}} \begin{bmatrix} (c_{13} - c_{23}) & 0 & 0 & [c_{13}(c_{22} + c_{12}) - c_{23}(c_{11} + c_{12})] & 0 & 0 \end{bmatrix}_k^{\mathrm{T}}, \tag{62}$$

so that the particular solution associated with $r\mathbf{g}_1$ is obtained from Eq. (C.5) as

$$\mathbf{q}_k(r) = [r_{k-1}\mathbf{P}(r/r_{k-1}) - r\mathbf{I}]z_1. \tag{63}$$

In the case of cylindrical orthotropy, $c_{11} = c_{22}$, Eq. (63) does not hold. Under this circumstance, an additional vector in the null space of $\mathbf{A}_k^{\mathrm{T}} - \mathbf{I}$ is $\begin{bmatrix} 1 - c_{12}/c_{11} & 0 & 0 & c_{11}^{-1} & 0 & 0 \end{bmatrix}_k^{\mathrm{T}}$ which is not orthogonal with $\mathbf{g}_1$. The particular solution is

$$\mathbf{q}_k(r) = [r_{k-1}\mathbf{P}(r/r_{k-1}) - r\mathbf{I}]\mathbf{h}_1 - [r_{k-1}\log r_{k-1}\mathbf{P}(r/r_{k-1}) - r\log r\mathbf{I}]\mathbf{h}_2, \tag{64}$$

where

$$\mathbf{h}_1 = \frac{\varepsilon(c_{13} + c_{23})}{4c_{11}} \begin{bmatrix} 1 & 0 & 0 & c_{12} - c_{11} & 0 & 0 \end{bmatrix}_k^{\mathrm{T}},$$

$$\mathbf{h}_2 = \frac{-\varepsilon(c_{13} - c_{23})}{2c_{11}} \begin{bmatrix} 1 & 0 & 0 & c_{12} + c_{11} & 0 & 0 \end{bmatrix}_k^{\mathrm{T}}.$$

For $\mathbf{f}_k(r) = r^2\mathbf{g}_2$ the particular solution is

$$\mathbf{q}_k(r) = [r_{k-1}^2\mathbf{P}(r/r_{k-1}) - r^2\mathbf{I}]\mathbf{z}_2, \tag{65}$$

where

$$\mathbf{z}_2 = (\mathbf{A}_k - 2\mathbf{I})^{-1}\mathbf{g}_2 = \frac{\vartheta Q_{24}}{c_{22} - 4c_{11}} \begin{bmatrix} 1 & 0 & 0 & (2c_{11} + c_{12}) & 0 & 0 \end{bmatrix}_k^{\mathrm{T}}, \quad (c_{22} \neq 4c_{11}).$$

For brevity we do not consider the rare case of the layer material with elastic constants $c_{22} = 4c_{11}$. Carrying out the multiplication in Eqs. (63)–(65) leads to

$$\mathbf{q}_k(r) = \begin{bmatrix} q_1(r) & 0 & 0 & q_4(r) & 0 & 0 \end{bmatrix}_k^{\mathrm{T}}. \tag{66}$$

With Eqs. (59) and (66), Eq. (35) is uncoupled to

$$\begin{bmatrix} U_1 \\ X_1 \end{bmatrix} = \begin{bmatrix} p_{11}(r/a) & p_{14}(r/a) \\ p_{41}(r/a) & p_{44}(r/a) \end{bmatrix} \begin{bmatrix} U_1(a) \\ X_1(a) \end{bmatrix} + \begin{bmatrix} q_1(r) \\ q_4(r) \end{bmatrix}, \tag{67}$$

$$\begin{bmatrix} V_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} r/a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_1(a) \\ W_1(a) \end{bmatrix} + \begin{bmatrix} s_{55}(r/a - a/r)/2 & -s_{56}(1 - r/a) \\ s_{56}(1 - r/a) & s_{66}\log(r/a) \end{bmatrix} \begin{bmatrix} Y_1(a) \\ Z_1(a) \end{bmatrix}, \tag{68}$$

$$Y_1(r) = (a/r)Y_1(a), \quad Z_1(r) = Z_1(a). \tag{69}$$

It can be seen that $Y_1(r)$ and $Z_1(r)$ cause axisymmetric deformations. Using Eqs. (46) and (50) along with Eq. (69) results in

$$\sigma_{r\theta}(r) = \tau_a(a/r)^2, \quad \sigma_{rz}(r) = s_a a/r. \tag{70}$$

This is a remarkable result in that the uniform shearing causes pure shears in the laminated tube of monoclinic cylindrically anisotropic materials, regardless of the elastic properties and the number of layers. Since uniform shearing does not produce $\sigma_{\theta z}$ and σ_z and is independent of A, B, C and ϑ, it is unrelated to P_z and M_t.

As a check of the validity of the approach, let us apply it to the plane deformation of a cylindrical anisotropic homogeneous tube under external pressure p_b. The solution of the problem can be found in Section 42 of Lekhnitskii's monograph (1981).

The boundary conditions on $r = a$ and b reduce to $X_1(a) = Y_1(a) = Z_1(a) = 0$ and $X_1(b) = -bp_b$, $Y_1(b) = Z_1(b) = 0$. Setting $\varepsilon = \vartheta = 0$ for the plane deformation, there follows from Eqs. (63)–(69)

$$q_1(r) = q_4(r) = V_1(r) = W_1(r) = Y_1(r) = Z_1(r) = 0. \tag{71}$$

With Eq. (71), the displacements and stresses are obtained as follows:

$$u_\theta = u_z = 0, \quad \sigma_{r\theta} = \sigma_{rz} = 0, \tag{72}$$

$$u_r = \frac{-bp_b}{1 - c^{2\kappa}}\left(\frac{1}{\kappa c_{11} + c_{12}}\rho^{\kappa} + \frac{1}{\kappa c_{11} - c_{12}}c^{2\kappa}\rho^{-\kappa}\right), \tag{73}$$

$$\sigma_r = \frac{-p_b}{1 - c^{2\kappa}}(\rho^{\kappa-1} - c^{2\kappa}\rho^{-\kappa-1}), \tag{74}$$

$$\sigma_\theta = \frac{-\kappa p_b}{1 - c^{2\kappa}}(\rho^{\kappa-1} + c^{2\kappa}\rho^{-\kappa-1}), \tag{75}$$

$$\sigma_z = \frac{-p_b}{1 - c^{2\kappa}}\left(\frac{\kappa c_{13} + c_{23}}{\kappa c_{11} + c_{12}}\rho^{\kappa-1} - \frac{\kappa c_{13} - c_{23}}{\kappa c_{11} - c_{12}}c^{2\kappa}\rho^{-\kappa-1}\right), \tag{76}$$

$$\sigma_{\theta z} = \frac{-p_b}{1 - c^{2\kappa}}\left(\frac{\kappa c_{14} + c_{24}}{\kappa c_{11} + c_{12}}\rho^{\kappa-1} - \frac{\kappa c_{14} - c_{24}}{\kappa c_{11} - c_{12}}c^{2\kappa}\rho^{-\kappa-1}\right), \tag{77}$$

where $c = a/b$, $\rho = r/b$, $(c \leqslant \rho \leqslant 1)$.

The above solution is in accordance with Lekhnitskii's solution. In Lekhnitskii's monograph the solution for the displacement was not obtained.

5. Bending

When the tube is subjected to a bending moment at the ends, the deformation and stress fields depend on θ. In view of the geometrical symmetry and cylindrical anisotropy, the deformation and stress at (r, θ) due to a bending moment about x_1-axis are equivalent to those at $(r, \theta + \pi/2)$ due to a bending moment of the same magnitude about x_2-axis. It follows that bending may be treated by considering either M_1 or M_2. The response due to bi-axial bending can be obtained by superposition. We consider the bending due to M_2 with which the constant A is associated.

The governing equation for bending due to M_2 is Eq. (25). For a laminated tube with specific layer properties the eigensolution of $\mathbf{A}_k$ can be easily obtained numerically. On forming the modal matrix, it is straightforward to determine the displacement and stress in the tube. To facilitate analysis we derive in the following the analytic expressions for the fundamental transfer matrix and the particular solution for cylindrically orthotropic materials.

For cylindrical orthotropy, $c_{14} = c_{24} = c_{34} = c_{56} = 0$, the eigenvalues of $\mathbf{A}_k$ are

$$\lambda_1 = 0, \quad \lambda_2 = \alpha, \quad \lambda_3 = \beta, \quad \lambda_4 = 0, \quad \lambda_5 = -\alpha, \quad \lambda_6 = -\beta, \tag{78}$$

where

$$\alpha = (1 + c_{22}/c_{11} - 2c_{12}/c_{11} + s_{55}Q_{22})^{1/2}, \qquad \beta = (c_{44}/c_{66})^{1/2}.$$

Using the eigenvectors the modal matrix is formed as

$$\mathbf{M}_k = \begin{bmatrix} -k_1 & \varphi_{21} & 0 & 0 & \varphi_{51} & 0 \\ k_1 & \varphi_{22} & 0 & \varphi_{42} & \varphi_{52} & 0 \\ 0 & 0 & \varphi_{33} & 0 & 0 & \varphi_{63} \\ 0 & k_2 & 0 & \varphi_{44} & k_2 & 0 \\ 0 & k_2 & 0 & \varphi_{45} & k_2 & 0 \\ 0 & 0 & \varphi_{36} & 0 & 0 & \varphi_{66} \end{bmatrix}_k, \tag{79}$$

where the expressions of k_1, k_2 and φ_{ij} are given in Appendix D.

By Eq. (C.3) the inverse of $\mathbf{M}_k$ is written out immediately as

$$\mathbf{M}_k^{-1} = \begin{bmatrix} \varphi_{44} & \varphi_{45} & 0 & 0 & -\varphi_{42} & 0 \\ k_2 & k_2 & 0 & -\varphi_{51} & -\varphi_{52} & 0 \\ 0 & 0 & \varphi_{66} & 0 & 0 & -\varphi_{63} \\ 0 & 0 & 0 & -k_1 & k_1 & 0 \\ -k_2 & -k_2 & 0 & \varphi_{21} & \varphi_{22} & 0 \\ 0 & 0 & -\varphi_{36} & 0 & 0 & \varphi_{33} \end{bmatrix}_k. \tag{80}$$

Indeed $\mathbf{M}_k\mathbf{M}_k^{-1} = \mathbf{I}$. Substituting Eqs. (79) and (80) in Eq. (38), replacing the diagonal matrix by the Jordan matrix for the repeated eigenvalues $\lambda_1 = \lambda_4 = 0$, we obtain

$$\mathbf{P}(r/r_{k-1}) = \begin{bmatrix} 1+p_{12} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & 1+p_{21} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{41} & 0 & 1+p_{54} & p_{45} & 0 \\ p_{41} & p_{41} & 0 & p_{54} & 1+p_{45} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{33} \end{bmatrix}_k, \tag{81}$$

where p_{ij} are functions of r/r_{k-1}, the expressions are given in Appendix D.

The non-homogeneous term in Eq. (25) is

$$\mathbf{f}_k(r) = r^2\mathbf{g}_k = r^2 A[-c_{13}/c_{11} \quad 0 \quad 0 \quad Q_{23} \quad Q_{23} \quad 0]_k^{\mathrm{T}}. \tag{82}$$

The matrix $\mathbf{A}_k - 2\mathbf{I}$ is *non-singular*, by Eq. (C.5), the particular solution is

$$\mathbf{q}_k(r) = [r_{k-1}^2\mathbf{P}(r/r_{k-1}) - r^2\mathbf{I}](\mathbf{A}_k - 2\mathbf{I})^{-1}\mathbf{g}_k. \tag{83}$$

From a mathematical point of view there may exist a combination of the elastic constants such that α and β in Eq. (78) are precisely equal to 2, making $\mathbf{A}_k - 2\mathbf{I}$ singular. However, this peculiar combination hardly occurs and does not warrant a special treatment.

To verify the state space approach we apply it to bending of a cylindrically orthotropic homogeneous tube with inner and outer radii being a and b. The solution can be found in Section 43 of Lekhnitskii's monograph (1981). In obtaining the analytic solution by Eqs. (81)–(83) the software *Mathematica* is used to advantage. The particular solution is obtained as

$$\mathbf{q}_2(r) = [\,\boldsymbol{\phi}_{u2}(r) \quad \boldsymbol{\phi}_{s2}(r)\,]^{\mathrm{T}} = -A\eta[a^2\mathbf{P}(r/a) - r^2\mathbf{I}][\,\varkappa \quad 1-\varkappa \quad 0 \quad 1 \quad 1 \quad 0]^{\mathrm{T}}, \tag{84}$$

where

$$\eta = -\frac{c_{66}[c_{23}(c_{11}+c_{12}) - c_{13}(c_{22}+c_{12})]}{c_{11}c_{22} - c_{12}^2 + c_{66}(c_{22} - 2c_{13} - 3c_{11})},$$

$$\varkappa = \frac{c_{13}c_{22} + c_{23}c_{12} + c_{66}(c_{23} - 3c_{13})}{2[c_{23}(c_{11}+c_{12}) - c_{13}(c_{22}+c_{12})]}.$$

The boundary condition (53) gives us

$$\mathbf{U}_2(a) = -\mathbf{T}_{su}^{-1}(b)\boldsymbol{\phi}_{s2}(b) = Aa^2\eta h[1 \quad 1 \quad 0]^{\mathrm{T}}, \tag{85}$$

where

$$h = \frac{[p_{41}(b/a) + p_{45}(b/a) + p_{54}(b/a)] - (b/a)^2 + 1}{2p_{41}(b/a)}.$$

Using Eqs. (84) and (85) in the formulation yields the displacements and stresses in the tube as follows:

$$\begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} = Ab^2 \begin{bmatrix} [c^2U(\rho) - (z/b)^2/2]\cos\theta \\ [c^2V(\rho) + (z/b)^2/2]\sin\theta \\ \rho(z/b)\cos\theta \end{bmatrix}, \tag{86}$$

$$[\sigma_r \quad \sigma_{r\theta}] = \frac{Abc^2}{\rho}[X(\rho)\cos\theta \quad Y(\rho)\sin\theta], \tag{87}$$

$$\begin{bmatrix} \sigma_\theta \\ \sigma_z \end{bmatrix} = \frac{Abc^2}{\rho}\left(\begin{bmatrix} Q_{22} & Q_{22} \\ Q_{23} & Q_{23} \end{bmatrix}\begin{bmatrix} U(\rho) \\ V(\rho) \end{bmatrix} + \begin{bmatrix} \hat{c}_{12} \\ \hat{c}_{13} \end{bmatrix}X(\rho) + (\rho/c)^2\begin{bmatrix} Q_{23} \\ Q_{33} \end{bmatrix}\right)\cos\theta, \tag{88}$$

$$\sigma_{rz} = \sigma_{\theta z} = 0, \tag{89}$$

where $c = a/b$, $\rho = r/b$, $(c \leqslant \rho \leqslant 1)$, and

$$U(\rho) = \eta\{h[1 + 2p_{12}(\rho/c)] - p_{12}(\rho/c) - p_{14}(\rho/c) - p_{15}(\rho/c) + \varkappa[(\rho/c)^2 - 1]\},$$

$$V(\rho) = \eta\{h[1 + 2p_{12}(\rho/c)] - p_{21}(\rho/c) - p_{24}(\rho/c) - p_{25}(\rho/c) + (\varkappa - 1)[(\rho/c)^2 - 1]\},$$

$$X(\rho) = \eta[2hp_{41}(\rho/c) - p_{41}(\rho/c) - p_{45}(\rho/c) - p_{54}(\rho/c) + (\rho/c)^2 - 1],$$

$$Y(\rho) = \eta[2hp_{41}(\rho/c) - p_{41}(\rho/c) - p_{45}(\rho/c) - p_{54}(\rho/c) + (\rho/c)^2 - 1].$$

The constants A is related to M_2 and can be determined through the end condition (10). The validity of the solution has been checked using *Mathematica* and is found in agreement with the stress expressions given by Lekhnitskii.

The approach is further examined by applying it to bending of a [0°/90°/0°] composite laminated tube composed of graphite/epoxy laminae, where 0° is the axial direction. The material constants used in the computation are $E_1 = 138$ GPa (20.0×10^6 psi), $E_2 = E_3 = 14.5$ GPa (2.1×10^6 psi), $G_{12} = G_{13} = G_{23} = 5.86$ GPa (0.85×10^6 psi), $\nu_{12} = \nu_{13} = \nu_{23} = 0.21$. The radii of each layer are $a = r_0 = 20$ mm, $r_1 = 30$ mm, $r_2 = 40$ mm, $b = r_3 = 50$ mm. An exact analysis on bending of a multilayered composite tube of a cylindrically anisotropic material is not found in the literature. Numerical results on the displacement and stress distribution can be easily computed following the present solution procedure using *Mathematica*. Figs. 3–7 show the displacements and the stresses at $z = 0$, $\theta = 0$ and 90° (where the displacement and stress are maximum) in the radial direction of the tube under a bending moment $M_2 = 1$ KN m. The radial and circumferential displacements u_r and u_θ are continuous through the thickness, the axial displacement u_z is zero at $z = 0$ for bending. The stress distribution indicates that the radial stress σ_r and the transverse shear stresses $\sigma_{r\theta}$ are continuous on the interfaces as expected. The axial stress σ_z is larger than the other stress components by an order of magnitude, indicating that the axial stress is dominant and the radial, the hoop

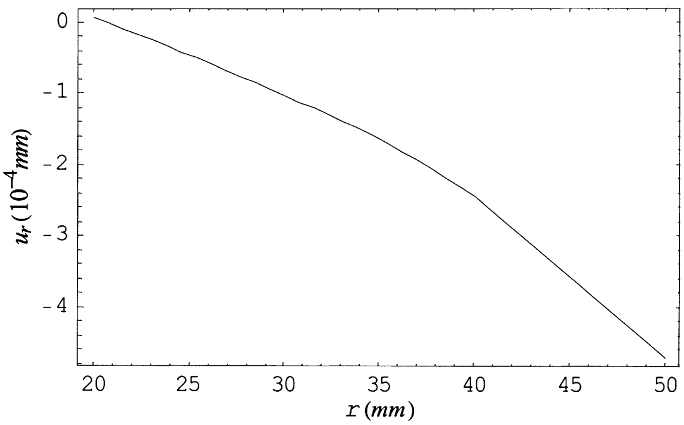

Fig. 3. Radial displacement u_r at $\theta = 0°$ through the thickness.

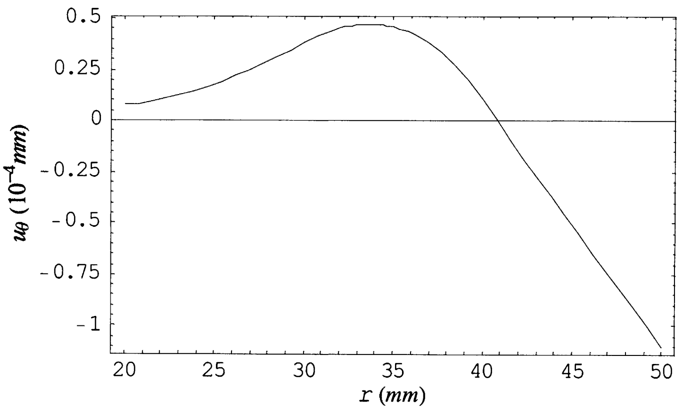

Fig. 4. Circumferential displacement u_θ at $\theta = 90°$ through the thickness.

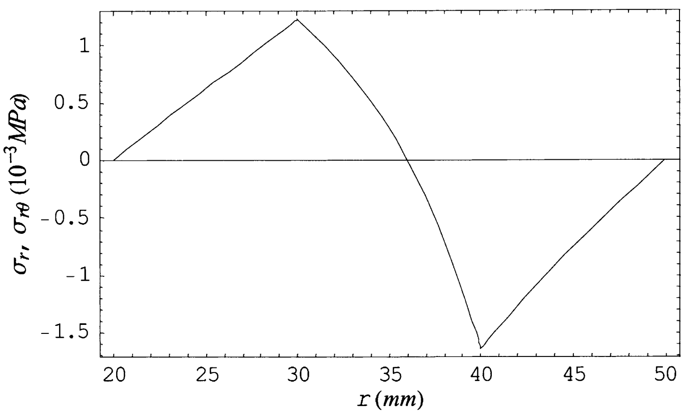

Fig. 5. Radial stress σ_r at $\theta = 0°$ and transverse shear stress $\sigma_{r\theta}$ at $\theta = 90°$ through the thickness.

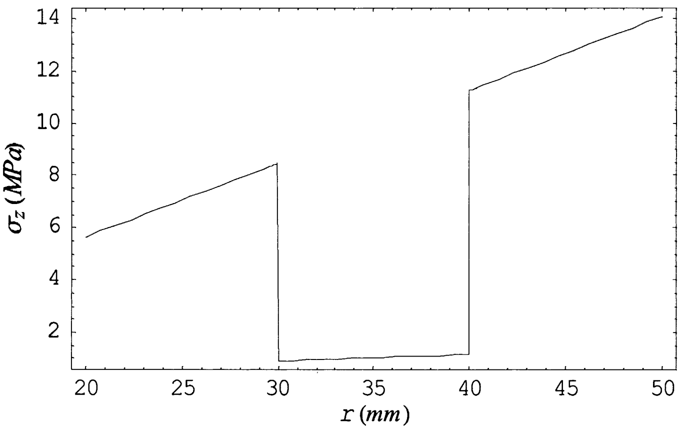

Fig. 6. Axial stress σ_z at $\theta = 0°$ through the thickness.

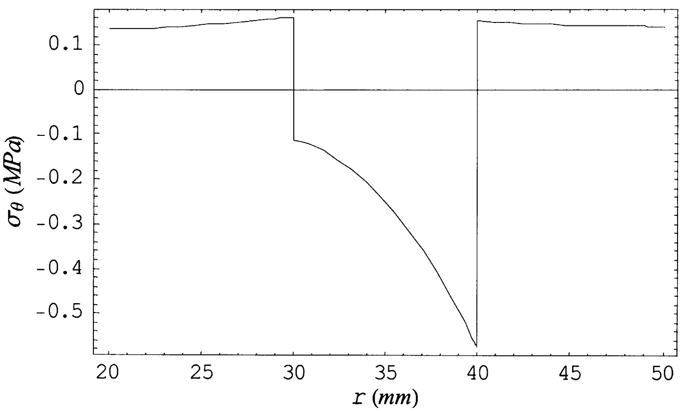

Fig. 7. Hoop stress σ_θ at $\theta = 0°$ through the thickness.

and the transverse shear stresses are secondary for bending of the tube. The axial stress is nearly linearly distributed through the thickness in each layer, with a jump in the 90° layer.

6. Relations between ε, ϑ, A, B and the applied loads

The solution for a general problem contains ε, ϑ, A and B that are related to the surface tractions and end loads. Since there is a one to one correspondence between them through Eqs. (7)–(10), these constants may be regarded as known a priori in the formulation. In practical situations, however, except for generalized plane deformation where $\varepsilon = \vartheta = A = B = 0$ may be specified in advance, normally the surface tractions and the end loads are prescribed; ε, ϑ, A and B are found after the stress due to a combined action of the applied loads is determined. An arbitrary set of ε, ϑ, A, B will not yield the stress for a combined action of prescribed loads. Hence it is necessary to establish the relations between ε, ϑ, A, B and the applied loads.

It has been shown that ε and ϑ are related to P_z and M_t, not to M_1 and M_2; A is related to M_2, B to M_2, and both are unrelated to P_z and M_t. Further, the internal and external pressure cause axisymmetric deformation, whereas the uniform shearing gives rise to a generalized plane deformation independent of ε, ϑ and the bending. It follows that extension, torsion and pressuring interact, but not coupled with uniform shearing and bending. Thus we may express the relations between ε, ϑ, A, B and the applied loads as

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \vartheta \end{bmatrix} + \begin{bmatrix} k_{13} & k_{14} \\ k_{23} & k_{24} \end{bmatrix} \begin{bmatrix} p_a \\ p_b \end{bmatrix} = \begin{bmatrix} P_z \\ M_t \end{bmatrix}, \tag{90}$$

$$k_{31}B = M_1, \quad k_{32}A = M_2, \tag{91}$$

where k_{ij} are the influence coefficients.

Obviously, k_{11} and k_{21} equal to P_z and M_t for $\varepsilon = 1$, $\vartheta = p_a = p_b = 0$; k_{12} and k_{22} equal to P_z and M_t for $\vartheta = 1$, $\varepsilon = p_a = p_b = 0$, and so on. Thus k_{11} and k_{21} can be found by prescribing $\varepsilon = 1$, $\vartheta = p_a = p_b = 0$ in the analysis to determine the P_z and M_t via Eqs. (7) and (8). Similarly, k_{12} and k_{22} can be found by prescribing $\vartheta = 1$, $\varepsilon = p_a = p_b = 0$ in the analysis to determine P_z and M_t, and so on. After obtaining k_{ij}, the constants ε, ϑ, A, B due to a combination of applied loads are determined from Eqs. (90) and (91) by a simple inversion.

7. Concluding remarks

The state space approach is an elegant and effective way of treating multilayered systems. In this paper we have formulated the problems of generalized plane strain, generalized torsion and bending of laminated composite tubes in a state space setting and developed a systematic method for stress analysis of the tube subjected to extension, torsion, bending, uniform pressuring and shearing. Other loading cases may be treated within the context provided that the loads do not vary axially and the end effect is neglected. We shall report the thermoelastic analysis of laminated composite tubes elsewhere (Tarn and Wang, 2001).

By means of the transfer matrix we have derived (48) which transmits the state vector from the inner surface to the outer surface. On $r = a$ and b either the traction or the displacement, or a mixed boundary conditions may be prescribed. Eq. (48) consists of six equations in six unknowns. Among the six components of $\mathbf{U}_i$ and $\mathbf{S}_i$ three are prescribed and the other three are unknown. When the tube is subjected to internal and external pressure, $\mathbf{S}_i$ are prescribed and $\mathbf{U}_i$ on $r = a$ and b are unknown. When displacements are prescribed, $\mathbf{U}_i$ are prescribed and $\mathbf{S}_i$ on $r = a$ and b are unknown. For a well-posed problem we can always find a unique solution for the unknowns following the same solution procedure. Once $\mathbf{U}_i$ and $\mathbf{S}_i$ on $r = a$ are found, the displacement and stress in the tube are determined via Eq. (45).

For the problem studied here the body force is absent. In some cases the body force does present. For instance, when the laminated tube is rotating at a constant angular velocity ω about its central axis, the centrifugal force constitutes a body force in the radial direction and produces an axisymmetric state. On introducing $R = \rho_k r\omega^2$ in Eq. (18), it can be shown that the matrix differential equation (29) takes the same form—only the vector $\mathbf{f}(r)$ needs to be modified—and the transfer matrix remains unchanged. The problem can be readily solved using the state space approach.

Acknowledgements

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Appendix A. Zero resultant shear forces

The resultant shear forces over a cross-section are

$$V_1 = \sum_{k=1}^{n} \int_{A_k} (\sigma_{rz}\cos\theta - \sigma_{\theta z}\sin\theta)_k \, \mathrm{d}A, \tag{A.1}$$

$$V_2 = \sum_{k=1}^{n} \int_{A_k} (\sigma_{rz}\sin\theta + \sigma_{\theta z}\cos\theta)_k \, \mathrm{d}A. \tag{A.2}$$

Expressing them in Cartesian coordinates using

$$\sigma_{rz} = \sigma_{xz}\cos\theta + \sigma_{yz}\sin\theta, \quad \sigma_{\theta z} = -\sigma_{xz}\sin\theta + \sigma_{yz}\cos\theta,$$

leads to

$$V_1 = \sum_{k=1}^{n} \int_{A_k} (\sigma_{xz})_k \, \mathrm{d}A, \quad V_2 = \sum_{k=1}^{n} \int_{A_k} (\sigma_{yz})_k \, \mathrm{d}A.$$

When the stress is independent of z, V_1 can be transformed to

$$\begin{aligned} V_1 &= \sum_{k=1}^{n} \int_{A_k} (\sigma_{xz})_k \, \mathrm{d}A = \sum_{k=1}^{n} \int_{A_k} [(x\sigma_{xz})_{,x} - x\sigma_{xz,x}]_k \, \mathrm{d}A = \sum_{k=1}^{n} \int_{A_k} [(x\sigma_{xz})_{,x} + (x\sigma_{yz})_{,y}]_k \, \mathrm{d}A \\ &= \sum_{k=1}^{n} \oint_{C_k} x(\sigma_{xz}n_x + \sigma_{yz}n_y)_k \, \mathrm{d}s, \end{aligned} \tag{A.3}$$

in which we have used the Stokes theorem and the equilibrium equation

$$\sigma_{xz,x} + \sigma_{yz,y} = 0.$$

The integrals of $x(\sigma_{xz}n_x + \sigma_{yz}n_y)_k$ and $x(\sigma_{xz}n_x + \sigma_{yz}n_y)_{k+1}$ cancel out along C_k, and the traction $\sigma_{xz}n_x + \sigma_{yz}n_y$ is uniform along the inner and outer contours, thus it can be taken out of the integral in Eq. (A.3). Set the origin at the centroid of the cross section, then $V_1 = 0$.

Similarly, it can be shown that

$$V_2 = \sum_{k=1}^{n} \oint_{C_k} y(\sigma_{xz}n_x + \sigma_{yz}n_y)_k \, \mathrm{d}s = 0.$$

Appendix B. 3D state equation

Using Eq. (1) we may express $u_{r,r}$ as

$$\frac{\partial u_r}{\partial r} = r^{-1}\begin{bmatrix} -\hat{c}_{12} & -(\hat{c}_{12}\partial_\theta + \hat{c}_{14}r\partial_z) & -(\hat{c}_{14}\partial_\theta + \hat{c}_{13}r\partial_z) \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} + c_{11}^{-1}\sigma_r, \tag{B.1}$$

where $\hat{c}_{ij} = c_{ij}/c_{11}$.

Substitution of Eq. (B.1) in Eqs. $(1)_2$, $(1)_3$, and $(1)_4$ gives us

$$\sigma_\theta = r^{-1}\begin{bmatrix} Q_{22} & Q_{22}\partial_\theta + Q_{24}r\partial_z & Q_{24}\partial_\theta + Q_{23}r\partial_z \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} + \hat{c}_{12}\sigma_r, \tag{B.2}$$

$$\sigma_z = r^{-1}[\,Q_{23} \quad Q_{23}\partial_\theta + Q_{34} r\partial_z \quad Q_{34}\partial_\theta + Q_{33} r\partial_z\,] \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} + \hat{c}_{13}\sigma_r, \tag{B.3}$$

$$\sigma_{\theta z} = r^{-1}[\,Q_{24} \quad Q_{24}\partial_\theta + Q_{44} r\partial_z \quad Q_{44}\partial_\theta + Q_{34} r\partial_z\,] \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} + \hat{c}_{14}\sigma_r, \tag{B.4}$$

where $Q_{ij} = c_{ij} - c_{1i}c_{1j}/c_{11}$.

By Eqs. $(1)_5$ and $(1)_6$, we have

$$\frac{\partial}{\partial r}\begin{bmatrix} u_\theta \\ u_z \end{bmatrix} = r^{-1}\begin{bmatrix} -\partial_\theta & 1 \\ -r\partial_z & 0 \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} + \begin{bmatrix} c_{66} & c_{56} \\ c_{56} & c_{55} \end{bmatrix}^{-1}\begin{bmatrix} \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix}. \tag{B.5}$$

Insertion of Eqs. (B.1) and (B.3) in Eqs. (15)–(17) yields

$$\frac{\partial}{\partial r}(r\sigma_r) = r^{-1}[\,Q_{22} \quad Q_{22}\partial_\theta + Q_{24} r\partial_z \quad Q_{24}\partial_\theta + Q_{23} r\partial_z\,] \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} + \hat{c}_{12}\sigma_r - \partial_\theta\sigma_{r\theta} - rR, \tag{B.6}$$

$$\frac{\partial}{\partial r}(r\sigma_{r\theta}) = -r^{-1}\partial_\theta[\,Q_{22} \quad Q_{22}\partial_\theta + Q_{24} r\partial_z \quad Q_{24}\partial_\theta + Q_{23} r\partial_z\,] \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} - \hat{c}_{12}\partial_\theta\sigma_r - \sigma_{r\theta} - r\Theta, \tag{B.7}$$

$$\frac{\partial}{\partial r}(r\sigma_{rz}) = -r^{-1}\partial_\theta[\,Q_{24} \quad Q_{24}\partial_\theta + Q_{44} r\partial_z \quad Q_{44}\partial_\theta + Q_{34} r\partial_z\,] \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} + \hat{c}_{14}\partial_\theta\sigma_r. \tag{B.8}$$

Casting Eqs. (B.1)–(B.8) in a matrix differential equation, we arrive at Eqs. (18) and (19).

Appendix C. Eigensolution of the matrix A

The eigenvalues λ_i and eigenvectors $\boldsymbol{\varphi}_i$ of $\mathbf{A}$ are determined from

$$\mathbf{A}\boldsymbol{\varphi}_i = \lambda_i\boldsymbol{\varphi}_i, \tag{C.1}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & -\mathbf{N}_1^{\mathrm{T}} \end{bmatrix},$$

$\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ are 3×3 matrices, and $\mathbf{N}_2 = \mathbf{N}_2^{\mathrm{T}}$, $\mathbf{N}_3 = \mathbf{N}_3^{\mathrm{T}}$.

The matrix $\mathbf{A}$ is similar to the Hamiltonian matrix (Zhong, 1995) such that $\mathbf{JAJ} = \mathbf{A}^{\mathrm{T}}$, where

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{0} \end{bmatrix},$$

$\mathbf{I}_3$ is a 3×3 identity matrix. The characteristics of the eigensolution of $\mathbf{A}$, useful to our purpose, are

1. if λ_i is an eigenvalue of $\mathbf{A}$ so is $-\lambda_i$
2. if $\boldsymbol{\varphi}_i$ is the eigenvector of $\mathbf{A}$ associated with λ_i, $\mathbf{J}^{\mathrm{T}}\boldsymbol{\varphi}_i$ is the eigenvector of $\mathbf{A}^{\mathrm{T}}$ associated with $-\lambda_i$
3. the eigenvectors $\boldsymbol{\varphi}_i$ of $\mathbf{A}$ possess the orthogonality property

$$\boldsymbol{\varphi}_i^{\mathrm{T}}\mathbf{J}\boldsymbol{\varphi}_{j+3} = \delta_{ij}, \quad (i,j = 1,2,3), \tag{C.2}$$

where the eigenvalues are set to be λ_i and $\lambda_{i+3} = -\lambda_i$, δ_{ij} is the Kronecker delta.

It follows that the six eigenvalues of $\mathbf{A}$ must be λ_1, λ_2, λ_3, and $\lambda_4 = -\lambda_1$, $\lambda_5 = -\lambda_2$, $\lambda_6 = -\lambda_3$. Let $\boldsymbol{\varphi}_i = [a_i \quad b_i \quad c_i \quad d_i \quad e_i \quad f_i]^{\mathrm{T}}$ be the eigenvector associated with λ_i, then

$$\mathbf{M} = \begin{bmatrix} \boldsymbol{\Phi}_a & \boldsymbol{\Psi}_a \\ \boldsymbol{\Phi}_b & \boldsymbol{\Psi}_b \end{bmatrix},$$

where

$$\boldsymbol{\Phi}_a = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad \boldsymbol{\Phi}_b = \begin{bmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{bmatrix}, \quad \boldsymbol{\Psi}_a = \begin{bmatrix} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{bmatrix}, \quad \boldsymbol{\Psi}_b = \begin{bmatrix} d_4 & d_5 & d_6 \\ e_4 & e_5 & e_6 \\ f_4 & f_5 & f_6 \end{bmatrix}.$$

Using Eq. (C.2) and its transpose along with $\mathbf{J}^{\mathrm{T}} = -\mathbf{J}$ yields

$$\begin{bmatrix} -\boldsymbol{\Psi}_a^{\mathrm{T}} & -\boldsymbol{\Psi}_b^{\mathrm{T}} \\ \boldsymbol{\Phi}_a^{\mathrm{T}} & \boldsymbol{\Phi}_b^{\mathrm{T}} \end{bmatrix} \mathbf{J} \begin{bmatrix} \boldsymbol{\Phi}_a & \boldsymbol{\Psi}_a \\ \boldsymbol{\Phi}_b & \boldsymbol{\Psi}_b \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & 0 \\ 0 & \mathbf{I}_3 \end{bmatrix}.$$

There follows

$$\mathbf{M}^{-1} = \begin{bmatrix} -\boldsymbol{\Psi}_a^{\mathrm{T}} & -\boldsymbol{\Psi}_b^{\mathrm{T}} \\ \boldsymbol{\Phi}_a^{\mathrm{T}} & \boldsymbol{\Phi}_b^{\mathrm{T}} \end{bmatrix} \mathbf{J} = \begin{bmatrix} \boldsymbol{\Psi}_b^{\mathrm{T}} & -\boldsymbol{\Psi}_a^{\mathrm{T}} \\ -\boldsymbol{\Phi}_b^{\mathrm{T}} & \boldsymbol{\Phi}_a^{\mathrm{T}} \end{bmatrix}. \tag{C.3}$$

The non-homogeneous terms in Eqs. (23), (25) and (27) may be written in short as

$$\mathbf{f}_k(r) = r^m \mathbf{g}, \quad m = 1, 2. \tag{C.4}$$

On substituting Eqs. (41) and (C.4) in Eq. (42) and making a change of variable $s = r/\xi$, Eq. (42) can be evaluated as

$$\mathbf{q}_k(r) = \left[\int_{r_{k-1}}^{r} \xi^{-1} (r/\xi)^{\mathbf{A}_k} \xi^m \, \mathrm{d}\xi \right] \mathbf{g} = \left[r^m \int_1^{r/r_{k-1}} s^{[\mathbf{A}_k - (m+1)\mathbf{I}]} \, \mathrm{d}s \right] \mathbf{g} = [r_{k-1}^m \mathbf{P}(r/r_{k-1}) - r^m \mathbf{I}] (\mathbf{A}_k - m\mathbf{I})^{-1} \mathbf{g}, \tag{C.5}$$

which holds when $\mathbf{A}_k - m\mathbf{I}$ is non-singular or $\mathbf{A}_k - m\mathbf{I}$ is singular but the vector in the null space (Pease, 1965) of $\mathbf{A}_k^{\mathrm{T}} - m\mathbf{I}$ is orthogonal with $\mathbf{g}$. When these conditions are violated, the particular solution must be modified using L'Hospital's rule.

Appendix D. Notations in bending analysis

The entries of the modal matrix (79) are

$$k_1 = [(c_{22} + c_{55})/(c_{11}c_{55})]^{1/2}/\alpha, \quad k_2 = [Q_{22}/(2\alpha)]^{1/2},$$

$$\varphi_{21} = k_2[c_{22} - (1+\alpha)c_{12}]/(\alpha c_{11} Q_{22}),$$

$$\varphi_{22} = k_2[(1+\alpha)c_{11} - c_{12} + c_{11}s_{55}Q_{22}]/(\alpha c_{11} Q_{22}),$$

$$\varphi_{33} = -\varphi_{63} = (c_{44}c_{66})^{-1/4}/\sqrt{2}, \quad \varphi_{36} = \varphi_{66} = (c_{44}c_{66})^{1/4}/\sqrt{2},$$

$$\varphi_{42} = k_1(c_{12} + c_{55})/(c_{22} + c_{55}), \quad \varphi_{44} = k_1[c_{12}^2 - c_{11}c_{22} - (c_{11} - c_{12})c_{55}],$$

$$\varphi_{45} = k_1(c_{22} - c_{12})c_{55}/(c_{22} + c_{55}), \quad \varphi_{51} = k_2[(1-\eta)c_{12} - c_{22}]/(\alpha c_{11}Q_{22}),$$

$$\varphi_{52} = k_2[(\alpha - 1)c_{11} + c_{12} - c_{11}s_{55}Q_{22}]/(\alpha c_{11}Q_{22}).$$

The entries of the fundamental transfer matrix (81) are

$$p_{12}(r/r_{k-1}) = k_2\varphi_{21}(r/r_{k-1})^{\alpha} - k_2\varphi_{51}(r/r_{k-1})^{-\alpha} - k_1\varphi_{44} - 1,$$

$$p_{14}(r/r_{k-1}) = -\varphi_{21}\varphi_{51}[(r/r_{k-1})^{\alpha} - (r/r_{k-1})^{-\alpha}] + k_1^2\log(r/r_{k-1}),$$

$$p_{15}(r/r_{k-1}) = -\varphi_{21}\varphi_{52}(r/r_{k-1})^{\alpha} + \varphi_{22}\varphi_{51}(r/r_{k-1})^{-\alpha} + k_1\varphi_{42} - k_1^2\log(r/r_{k-1}),$$

$$p_{21}(r/r_{k-1}) = k_2\varphi_{22}(r/r_{k-1})^{\alpha} - k_2\varphi_{52}(r/r_{k-1})^{-\alpha} + k_1\varphi_{44},$$

$$p_{24}(r/r_{k-1}) = -\varphi_{22}\varphi_{51}(r/r_{k-1})^{\alpha} + \varphi_{21}\varphi_{52}(r/r_{k-1})^{-\alpha} - k_1\varphi_{42} - k_1^2\log(r/r_{k-1}),$$

$$p_{25}(r/r_{k-1}) = -\varphi_{22}\varphi_{52}[(r/r_{k-1})^{\alpha} - (r/r_{k-1})^{-\alpha}] + k_1^2\log(r/r_{k-1}),$$

$$p_{33}(r/r_{k-1}) = [(r/r_{k-1})^{\beta} + (r/r_{k-1})^{-\beta}]/2,$$

$$p_{36}(r/r_{k-1}) = (c_{44}c_{55})^{-1/2}[(r/r_{k-1})^{\beta} - (r/r_{k-1})^{-\beta}]/2,$$

$$p_{41}(r/r_{k-1}) = (Q_{22}/2\alpha)[(r/r_{k-1})^{\alpha} - (r/r_{k-1})^{-\alpha}],$$

$$p_{45}(r/r_{k-1}) = -k_2\varphi_{52}(r/r_{k-1})^{\alpha} + k_2\varphi_{22}(r/r_{k-1})^{-\alpha} + k_1\varphi_{44},$$

$$p_{54}(r/r_{k-1}) = -k_2\varphi_{51}(r/r_{k-1})^{\alpha} + k_2\varphi_{21}(r/r_{k-1})^{-\alpha} - k_1\varphi_{45},$$

$$p_{63}(r/r_{k-1}) = (c_{44}c_{55})^{1/2}[(r/r_{k-1})^{\beta} - (r/r_{k-1})^{-\beta}]/2.$$

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